# TEMPERATURE FIELD IN A HOLLOW CYLINDER WITH INTERNAL PERIODIC INSTANTANEOUS HEAT SOURCES AND THE ROOTS OF THE CORRESPONDING CHARACTERISTIC EQUATION

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Inzhenerno-Fizicheskii Zhurnal, Vol. 13, No. 3, pp. 326–329, 1967

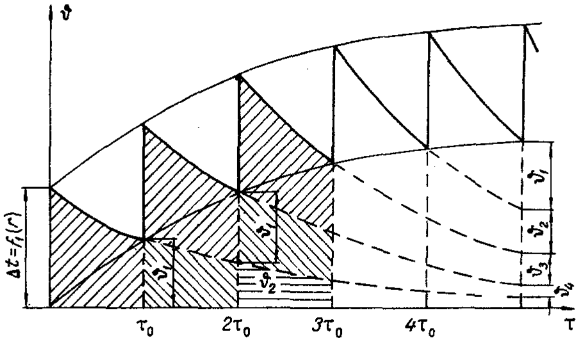
UDC 536.2.01

A solution is obtained for the temperature field in the active element (in the form of a hollow cylinder) of a pulsed laser with convection at the inside and outside surfaces. It is shown that for existing lasers, the solution may be truncated after the first term of the series. Values of the first root of the characteristic equation are obtained for Biot numbers ranging from 1 to 100.

Studying the thermal conditions in a hollow cylinder with convective heat transfer at both surfaces, one arrives at a characteristic equation of the form

$$[xJ_1(x) + AJ_0(x)]\,[kxY_1(kx) - BY_0(kx)] - \\ - [xY_1(x) + AY_0(x)]\,[kxJ_1(kx) - BJ_0(kx)] = 0. \qquad (1)$$

This work was performed to determine the temperature field in the hollow cylindrical active element of a pulsed laser.




Heating diagram for the active element of a laser.

We know that the operation of the active element of a pulsed laser involves a sequence of cycles which consist of a pumping period (powerful gas-filled lamp discharge followed by a pause (cooling period). Conversion of a portion of the pumping radiation absorbed by the active element into heat occurs during the pumping periods, the pumping pulse repetition rate being $10^2$ to $10^5$. In view of this, the problem of the temperature field in the active elements of a laser within the limits of one cycle, can be reduced to the following problem:

$$\frac{\partial \vartheta_m(r,\tau)}{\partial r} = a\left[\frac{\partial^2 \vartheta_m(r,\tau)}{\partial r^2} + \frac{1}{r}\,\frac{\partial \vartheta_m(r,\tau)}{\partial r}\right], \qquad (a)$$

$$\vartheta_m(r,0) = f(r) + \vartheta_{m-1}(r,\tau_0), \qquad (b)$$

$$\frac{\partial \vartheta_m(r_0,\tau)}{\partial r} - \frac{\alpha_1}{\lambda}\vartheta_m(r_0,\tau) = 0, \qquad (c)$$

$$\frac{\partial \vartheta_m(R,\tau)}{\partial r} + \frac{\alpha_2}{\lambda}\vartheta_m(R,\tau) = 0. \qquad (d)$$

The solution of system (a)–(d) is known (see, for example, [1, 2]) to have the form

$$\vartheta_m(r,\tau) = \sum_{i=1}^{\infty} E_i W_0(p_i r)\exp(-ap_i^2\tau), \qquad (2)$$

where $p_i$ denotes the roots of Eq. (1), (in which $x = pr_0$), $W_0(p_i r)$ is a linear combination of Bessel functions of the form

$$W_0(p_i r) = -\left[\frac{\alpha_1}{\lambda}Y_0(p_i r_0) + p_i Y_1(p_i r_0)\right]J_0(p_i r) + \\ + \left[p_i J_1(p_i r_0) + \frac{\alpha_1}{\lambda}J_0(p_i r_0)\right]Y_0(p_i r), \qquad (3)$$

and the numbers $E_i$ are the series coefficients of the distribution function $f(r)$ of the absorbed pumping radiation and temperature toward the end of the (m − 1)-th period,

$$f(r) + \vartheta_{m-1}(r,\tau_0) = \sum_{i=1}^{\infty} E_i W_0(p_i r). \qquad (4)$$

We have to determine $\vartheta_{m-1}(r,\tau_0)$.

Within the limits of the linear problem, it may be shown (as in [3–5]) that to determine the thermal conditions in the laser active element during any cycle, it is sufficient to know the temperature field during the first cycle. This follows from the fact that the temperature field, for example during the second cycle, can be regarded as the superposition of two fields: the temperature field regenerated by the second pumping pulse—identical to the temperature distribution during the time interval $[0, \tau_0]$—and the residual temperature field of the first pulse during the time interval $[\tau_0, 2\tau_0]$ (see figure).

Hence, it can be readily shown that the final form of the solution of system (a)–(d) for the m-th cycle is

$$\vartheta_m(r,\tau) = \sum_{i=1}^{\infty} E_i W_0(p_i r) \times \\ \times \frac{1-\exp(-map_i^2\tau_0)}{1-\exp(-ap_i^2\tau_0)}\exp(-ap_i^2\tau). \qquad (5)$$

We shall examine the well-known solution for an active element in the form of a solid circular cylinder [3–5], postulating that k = 0.

Estimates of the convergence of series (5), obtained by a method described in [6], show that for

Roots of the Characteristic Equation (1) for Various Values of k

| *A* | *B* | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 15 | 25 | 50 | 100 |
| | | | | | | | $k = 1.5$ | | | | | | | |
| 1 | 1.78 | 2.08 | 2.31 | 2.51 | 2.67 | 2.79 | 2.89 | 2.97 | 3.03 | 3.09 | 3.30 | 3.50 | 3.59 | 3.59 |
| 2 | 2.04 | 2.33 | 2.57 | 2.78 | 2.95 | 3.07 | 3.17 | 3.25 | 3.31 | 3.37 | 3.59 | 3.82 | 3.99 | 4.07 |
| 3 | 2.24 | 2.54 | 2.78 | 2.99 | 3.16 | 3.29 | 3.40 | 3.48 | 3.55 | 3.61 | 3.83 | 4.06 | 4.24 | 4.33 |
| 5 | 2.49 | 2.80 | 3.04 | 3.25 | 3.42 | 3.55 | 3.67 | 3.76 | 3.84 | 3.90 | 4.15 | 4.39 | 4.59 | 4.67 |
| 10 | 2.73 | 3.05 | 3.30 | 3.52 | 3.69 | 3.82 | 3.94 | 4.04 | 4.12 | 4.18 | 4.43 | 4.73 | 5.05 | 5.17 |
| 50 | 3.20 | 3.53 | 3.81 | 4.04 | 4.22 | 4.36 | 4.50 | 4.60 | 4.69 | 4.78 | 5.06 | 5.35 | 5.70 | 5.83 |
| | | | | | | | $k = 2$ | | | | | | | |
| 1 | 1.06 | 1.28 | 1.44 | 1.55 | 1.63 | 1.69 | 1.74 | 1.78 | 1.81 | 1.83 | 1.93 | 2.01 | 2.06 | 2.09 |
| 2 | 1.20 | 1.43 | 1.59 | 1.70 | 1.78 | 1.84 | 1.89 | 1.93 | 1.97 | 2.00 | 2.11 | 2.21 | 2.27 | 2.31 |
| 3 | 1.30 | 1.58 | 1.70 | 1.81 | 1.89 | 1.95 | 2.01 | 2.05 | 2.09 | 2.12 | 2.23 | 2.33 | 2.39 | 2.43 |
| 5 | 1.37 | 1.66 | 1.79 | 1.91 | 1.99 | 2.06 | 2.12 | 2.16 | 2.20 | 2.24 | 2.36 | 2.48 | 2.56 | 2.60 |
| 10 | 1.53 | 1.83 | 1.97 | 2.09 | 2.17 | 2.24 | 2.30 | 2.34 | 2.38 | 2.42 | 2.54 | 2.66 | 2.75 | 2.79 |
| 50 | 1.65 | 1.96 | 2.11 | 2.23 | 2.31 | 2.38 | 2.45 | 2.50 | 2.54 | 2.58 | 2.70 | 2.83 | 2.93 | 2.99 |
| | | | | | | | $k = 3$ | | | | | | | |
| 1 | 0.64 | 0.77 | 0.85 | 0.91 | 0.95 | 0.98 | 1.00 | 1.02 | 1.04 | 1.05 | 1.09 | 1.13 | 1.16 | 1.16 |
| 2 | 0.68 | 0.82 | 0.92 | 0.98 | 1.03 | 1.06 | 1.09 | 1.11 | 1.13 | 1.14 | 1.19 | 1.23 | 1.26 | 1.26 |
| 3 | 0.71 | 0.85 | 0.96 | 1.02 | 1.07 | 1.10 | 1.13 | 1.15 | 1.17 | 1.19 | 1.24 | 1.28 | 1.32 | 1.32 |
| 5 | 0.75 | 0.89 | 1.00 | 1.07 | 1.12 | 1.15 | 1.18 | 1.20 | 1.22 | 1.24 | 1.29 | 1.34 | 1.38 | 1.38 |
| 10 | 0.77 | 0.92 | 1.03 | 1.10 | 1.15 | 1.19 | 1.22 | 1.25 | 1.27 | 1.29 | 1.34 | 1.40 | 1.44 | 1.44 |
| 15 | 0.81 | 0.98 | 1.08 | 1.15 | 1.20 | 1.24 | 1.27 | 1.30 | 1.32 | 1.34 | 1.39 | 1.45 | 1.50 | 1.50 |
| | | | | | | | $k = 4$ | | | | | | | |
| 1 | 0.46 | 0.55 | 0.60 | 0.64 | 0.67 | 0.69 | 0.71 | 0.72 | 0.73 | 0.74 | 0.77 | 0.79 | 0.80 | 0.80 |
| 2 | 0.50 | 0.60 | 0.65 | 0.70 | 0.73 | 0.75 | 0.77 | 0.78 | 0.79 | 0.80 | 0.83 | 0.85 | 0.86 | 0.86 |
| 3 | 0.51 | 0.62 | 0.67 | 0.72 | 0.75 | 0.77 | 0.79 | 0.80 | 0.81 | 0.82 | 0.86 | 0.89 | 0.91 | 0.91 |
| 5 | 0.52 | 0.64 | 0.70 | 0.75 | 0.78 | 0.80 | 0.82 | 0.83 | 0.84 | 0.85 | 0.89 | 0.92 | 0.94 | 0.94 |
| 10 | 0.53 | 0.66 | 0.72 | 0.77 | 0.80 | 0.82 | 0.84 | 0.85 | 0.86 | 0.87 | 0.91 | 0.95 | 0.97 | 0.97 |
| 50 | 0.54 | 0.67 | 0.73 | 0.78 | 0.82 | 0.84 | 0.86 | 0.87 | 0.88 | 0.89 | 0.93 | 0.97 | 0.99 | 1.00 |
| | | | | | | | $k = 5$ | | | | | | | |
| 1 | 0.35 | 0.41 | 0.46 | 0.49 | 0.51 | 0.53 | 0.55 | 0.56 | 0.57 | 0.58 | 0.60 | 0.61 | 0.61 | 0.61 |
| 2 | 0.36 | 0.42 | 0.47 | 0.50 | 0.53 | 0.56 | 0.58 | 0.60 | 0.61 | 0.62 | 0.64 | 0.65 | 0.65 | 0.65 |
| 3 | 0.38 | 0.44 | 0.48 | 0.52 | 0.55 | 0.58 | 0.60 | 0.62 | 0.63 | 0.64 | 0.66 | 0.67 | 0.67 | 0.67 |
| 5 | 0.42 | 0.48 | 0.52 | 0.55 | 0.58 | 0.60 | 0.62 | 0.64 | 0.65 | 0.66 | 0.68 | 0.69 | 0.70 | 0.70 |
| 10 | 0.46 | 0.52 | 0.55 | 0.58 | 0.61 | 0.63 | 0.65 | 0.66 | 0.67 | 0.68 | 0.70 | 0.71 | 0.72 | 0.72 |
| 50 | 0.51 | 0.57 | 0.60 | 0.62 | 0.64 | 0.65 | 0.66 | 0.67 | 0.68 | 0.69 | 0.71 | 0.73 | 0.74 | 0.75 |

actual lasers (Biot number > 0.05, Fourier number > > 0.05) the computation may be limited to the first term of the series. However, practical realization of a solution of (5) is greatly complicated even then, since the values of the roots of the characteristic equation (1) are unknown.

To obtain numerical values of these roots, use was made of a graph-analytic method, described in [7], in a form extended to the present case.

The values of B for various x are obtained by writing (1) in the following form:

$$B = \{[xY_1(x) + AY_0(x)]\, kxJ_1(kx) - [xJ_1(x) + AJ_0(x)]\, kxY_1(kx)\} \times \{[xY_1(x) + AY_0(x)]\, J_0(kx) - [xJ_1(x) + AJ_0(x)]\, Y_0(kx)\}^{-1} \quad (6)$$

and varying systematically the values of A and k.

The values required are taken from a family of curves plotted from (6) in a system of coordinates B, x. The results are given in the table. The ordinates of the points of intersection of curves (6) with the ordinate axis (B = 0) agree with the results obtained in [7], and are omitted here. To obtain results useful for actual lasers, the values of k were taken as 1, 1.5, 2, 3, 4, and 5, and the values of A as 1, 2, 3, 5, 10, and 50. The absolute error of the values tabulated is two to three units in the final digit.

## NOTATION

$\vartheta(r, \tau)$ is the excess temperature at a point on radius r at an instant of time $\tau$ relative to the ambient temperature; A and B are the Biot numbers—$A = \alpha_1 r_0/\lambda$ and $B = \alpha_2 R/\lambda$, respectively; $\alpha_1$ and $\alpha_2$ are heat-transfer coefficients at the inner cylinder surface of radius $r_0$ and at the outer surface of radius $R = kr_0$, respectively; $\lambda$ is the thermal conductivity; $a$ is the thermal diffusivity.

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23 November 1966

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